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The Ellipsograph of Proclus.[*]

By E. M. Blake.

If a plane σ containing two points E and E_1 moves upon a coincident plane σ_1 containing two straight lines g and g_1 so that E remains upon g and E_1 upon g_1, the two planes form a mechanism possessing the following well-known properties: Every point of σ traces an ellipse upon σ_1, and every point of σ_1 traces a limaçon upon σ.[†] A circle c of radius a in σ rolls upon the inner side of a circle c_1 of radius $2a$ in σ_1. Every point of c describes a straight line passing through the center of c_1. Any two of these lines, with the points which generate them, can be taken for g, g_1 and E, E_1 in defining the movement.

The object of the present paper is to make a brief study of—1° the curves generated by the points of σ and σ_1; 2° the ruled surfaces generated by any straight line carried by σ or σ_1 and not parallel to them; 3° the curves enveloped by any straight line of σ or σ_1; 4° the developables enveloped by carried planes.

The point loci have been given by Cayley[‡] and Schell.[§] The line loci for the ellipsograph have been determined by Burmester,[||] although he omits their equation, but those for the inverse movement are believed to be given here for the first time. The envelope of the straight line of σ which passes through E and E_1 when g and g_1 are at right angles, was found by Chasles to be the four-cusped hypocycloid.[*] The only other lines of σ, the equations of whose envelopes have been previously determined, are those forming a square with EE_1.[†]

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[*] Read before the Chicago Section of the American Mathematical Society, April 9, 1898, at which eight thread models of surfaces described in the paper were exhibited. This embodies the author's paper, "Upon a Ruled Surface of the Fourth Order Mechanically Generated," read before the Society December 31, 1897. Paper revised, Ithaca, January, 1900.

[†] The discovery of the first property is accredited to Proclus; Chasles, "Aperçu historique," 2d ed., p. 49. The well-known chuck for turning figures with elliptical cross-sections, invented by Leonardo da Vinci, is an application of the mechanism. For other historical notes, see Burmester. "Lehrbuch der Kinematik I," Leipzig, 1888, pp. 36-42; A. v. Braunmühl, "Studie über Curvenerzeugung" in the "Katalog mathematischer Modelle," by Dyck.

[‡] "On the Kinematics of a Plane." Quarterly Journal, XVI, 1878, pp. 1-8.

[§] "Theorie der Bewegung und Kräfte I, pp. 227-230.

[||] "Kinematische Flächen erzeugung vermittelst cylindrischer Rollung." Zeit. für Math. u. Phys. XXXIII, 1888, pp. 337-348. The surface with real and distinct nodal straight lines is also given by Mannheim, Comptes Rendus de l'Acad. de Paris, LXXVI, 1873, pp. 635-639; Bulletin de la Soc. Math. de France, I, 1875, pp. 106-114. Menzel, "Ueber die Bewegung einer starren Geraden." Dissertation, Münster, 1891.

Point and Straight Line Loci of the Ellipsograph.

Denoting by (x, y) the rectangular coordinates of any point of σ, taking for origin the center O of c, and by (x_1, y_1) the same for any point of σ_1 taking for origin the center O_1 of c_1; the equation

$$x_1^2(x^2-2ax+a^2+y^2)-4ax_1y_1y+y_1^2(x^2+2ax+a^2+y^2)=(x^2+y^2-a^2)^2 \qquad (1)$$

represents the ellipse in σ_1 traced by the point (x, y) of σ or the limaçon traced by (x_1, y_1) upon σ.[‡]

From the equation we infer the following properties of the congruence of ellipses described by the points of σ: They all have the point O_1 for center, and rotation through half the angle whose tangent is $\frac{y}{x}$ brings their axes into coincidence with the coordinate axes. The points of a circumference whose center is O describe congruent ellipses, which degenerate to the diameters of c_1 when the circumference is c. The point O describes a circle of radius a. The points of a straight line through O describe ellipses whose axes lie upon two fixed lines.[§]

To obtain the equation of the surfaces generated by the straight lines carried by σ and oblique to it, we proceed as in a former paper.[‖] Take axes of z and z_1 passing respectively through O and O_1 and perpendicular to σ and σ_1. Let l be a representative generator parallel to yOz, passing through the point

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[*] "Aperçu historique," p. 69. The author also remarks that the envelope of any other straight line is the involute of a hypocycloid.

[†] J. B. Pomey, "Enveloppes des côtés d'un carré invariable dont deux sommets décrivent deux droites rectangulaires." Nouvelles Annales (3), V, 1886, pp. 520–530.

[‡] Cayley and Schell, loc. cit.

[§] For greater detail see J. S. et N. Vaněček. "Sur les ellipses décrites par les points invariablements liés à un segment constant et sur une surface circulaire du huitième ordre." Bulletin de la Soc. Math. de France, XI, 1883, pp. 76–88.

[‖] American Journal, XXI, 1899, pp. 260–261.

$(p, 0)$ of σ and making with that plane the angle whose cotangent is s; its locus is

$$x_1^2(p^2 - 2ap + a^2 + s^2 z_1^2) - 4as\,x_1 y_1 z_1 + y_1^2(p^2 + 2ap + a^2 + s^2 z_1^2) = (p^2 + s^2 z_1^2 - a^2)^2 \quad (2)$$

obtained from (1) by substituting for x and y respectively p and sz_1.

There is no difficulty now in establishing the following theorem: *Any straight line oblique to σ and carried by it generates a quartic scroll having a real isolated double generator in the plane σ_1 at infinity and two other nodal straight lines which are: either real and distinct, coincident, or imaginary according as l' (the orthogonal projection of l upon σ) intersects c in two real, two coincident, or two imaginary points.*

Point and Straight Line Loci of the Inverse of the Ellipsograph.

Regarding the plane σ as fixed, the points (x_1, y_1) of σ_1 describe upon it a congruence of limaçons, defined by equation (1) and possessing the following properties: Any point on c_1 describes a cardioide, any point within a limaçon having a node, and any point without one having a conjugate point. All points of a circumference with center at O_1 trace congruent curves. They may be brought into coincidence by rotation about O through twice the angle whose tangent is $\frac{y_1}{x_1}$. The point O_1 describes the circle c twice during a cycle of the movement. The real double points of all the curves are upon c. The points of a line through O_1 making the angle θ with the axis of x_1 generate curves with the same line of symmetry making the angle 2θ with Ox.

As in the preceding case, we have for the locus of l the equation

$$p^2(x^2 - 2ax + a^2 + y^2) - 4apsyz + s^2 z^2(x^2 + 2ax + a^2 + y^2) = (x^2 + y^2 - a^2)^2, \quad (3)$$

and the following theorem: *Any straight line oblique to σ_1 and carried by it generates a quartic scroll having: if l' passes through O_1, a nodal circle in σ (or a parallel plane) and an intersecting nodal straight line; and if l' does not pass through, a nodal cubical ellipse. Upon the latter are two imaginary pinch-points, and in addition either two real, two coincident, or two imaginary pinch-points according as l' intersects c_1 in two real, two coincident, or two imaginary points.*

Singularities of Point Loci.

The circle of inflexions* for the ellipsograph is c, and hence, any point upon it is always describing an inflexion upon its trajectory, i. e., the trajectory is a straight line. The circle of inflexions for the inverse movement has the same radius as c, and passes, during a complete cycle of the movement, once around c_1, remaining tangent to it and to the concentric circle of radius $4a$. Hence, the limaçon generated by a point of σ_1 between these two circumferences has two real points of inflexion. The two inflexions merge into one of the next higher order if the tracing-point be taken upon the circumference of radius $4a$.

The movement of a plane σ upon another σ_1 is either a rotation about a fixed point, a movement such that all straight lines of σ remain continually parallel to their initial positions, or the result of rolling a curve C of σ upon a curve G_1 of σ_1. Movements of the latter type are the only ones which give rise to any but trivial problems in loci and envelopes. It is easy to show that *the movement defined by the ellipsograph is the only movement of the last type which causes all points of σ to describe conics.* For any point of C describes a curve with at least one cusp. A conic having a cusp consists of two coincident straight lines. Hence, at least two points of σ describe straight lines, which defines the movement of the ellipsograph.

The Curves Enveloped by Straight Lines carried Parallel to the Plane of Movement, and the Developables Enveloped by carried Planes.

We will commence by finding the envelope of the line $y = d$ carried by σ. Assume for convenience $a = \frac{1}{2}$, then the equation of one of the positions of $y = d$ in σ_1 is

$$\frac{x_1}{b'} + \frac{y_1}{b} = 1 + \frac{d}{bb'}\,; \quad b^2 + b'^2 = 1,$$

b and b' being respectively the intercepts of $y = 0$ upon the axes of x_1 and y_1. Eliminating b', we have

$$b^4 - 2y_1\,b^3 + (x_1^2 + y_1^2 - 1)\,b^2 + 2\,(y_1 - dx_1)\,b + d^2 - y_1^2 = 0.$$

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* Koenigs, "Leçons de cinématique," I, pp. 144–154; Schoenflies, "La géométrie du mouvement." Paris, 1893, Chapter I.

The discriminant of this with respect to b equated to zero gives

$$
\begin{aligned}
&(x_1^2 + y_1^2)^3 - (3 + d^2)\, y_1^4 - 18 d x_1 y_1^3 + (21 - 2d^2)\, x_1^2 y_1^2 - 18 d x_1^3 y_1 - (3 + d^2)\, x_1^4 \\
&\quad + (3 + 20d^2)\, y_1^2 - (36d - 16d^3)\, x_1 y_1 + (3 + 20d^2)\, x_1^2 - 16d^4 + 8d^2 - 1 = 0, \qquad (4)
\end{aligned}
$$

the envelope required, and $(x_1 y_1 - d)^2 = 0$. The latter may be regarded as the locus of nodes, which should occur twice. It is due to the enveloping line having two positions for any value of b.

The equation (4) represents, except for a rotation about the origin, the curve enveloped by any straight line of σ. It also represents with the same reservation the cylinder enveloped by any plane carried by, and perpendicular to, σ. The surface enveloped by any plane making an acute angle with σ is evidently congruent to one enveloped by a plane passing through $y = 0$ and making the same angle with σ. They may be brought into coincidence by rotation about and translation along the axis of z_1. We proceed immediately to the determination and discussion of the equation of the latter surface, in the course of which we shall be in a position to determine the character of its plane sections parallel to σ_1, which are the same as the curves (4).

The equation of the surface enveloped by the plane which passes through $y = 0$ and makes with σ the angle whose cotangent is s, is obtained from (4) by substituting sz_1 for d. The result is

$$
\begin{aligned}
&(x_1^2 + y_1^2)^3 - (3 + s^2 z_1^2)\, y_1^4 - 18 s x_1 y_1^3 z_1 + (21 - 2s^2 z_1^2)\, x_1^2 y_1^2 \\
&\quad - 18 s x_1^3 y_1 z_1 - (3 + s^2 z_1^2)\, x_1^4 + (3 + 20 s^2 z_1^2)\, y_1^2 - (36 s z_1 - 16 s^3 z_1^3)\, x_1 y_1 \\
&\quad + (3 + 20 s^2 z_1^2)\, x_1^2 - 16 s^4 z_1^4 + 8 s^2 z_1^2 - 1 = 0. \qquad (5)
\end{aligned}
$$

Rotating the coordinate axes through 45° about the axis of z_1, gives this equation the more convenient form

$$
\begin{aligned}
&(x_1^2 + y_1^2)^3 - (3 + s^2 z_1^2)(x_1^2 + y_1^2)^2 + \tfrac{27}{4}(x_1^2 - y_1^2)^2 - 9 s z_1 (x_1^4 - y_1^4) \\
&\qquad + (3 + 20 s^2 z^2)(x_1^2 + y_1^2) - (18 s z_1 - 8 s^3 z_1^3)(x_1^2 - y_1^2) - 16 s^4 z_1^4 \\
&\qquad + 8 s^2 z_1^2 - 1 = 0, \qquad (6)
\end{aligned}
$$

to which the discussion which follows will apply.

The surface is of the sixth order and symmetrical with respect to $x_1 = 0$ and $y_1 = 0$. Its intersection with $y_1 = 0$ consists of the two straight lines $z_1 = \pm \dfrac{x_1}{s} - \dfrac{1}{2s}$ and the parabola $x_1^2 - 4 s z_1 + 2 = 0$ taken twice. The lines are tangent to the parabola at $\left(\pm 2, \dfrac{3}{2s}\right)$. The section by $x_1 = 0$ consists of the lines $z_1 = \pm \frac{y_1}{s} + \frac{1}{2s}$ and the parabola $y_1^2 + 4z_1 + 2 = 0$ taken twice. The lines are tangent to the parabola at $\left(\pm 2, -\frac{3}{2s}\right)$. The two parabolas are nodal lines of the surface. Each has a segment lying upon the surface and an isolated segment. The former contains the vertex of the parabola, and is included between its points of tangency with the straight lines mentioned above.

The edge of regression is readily verified to be the intersection of the two cylinders

$$\left.\begin{aligned} x_1^2 &= \tfrac{4}{27}\left(sz_1 + \tfrac{3}{2}\right)^3, \\ y_1^2 &= -\tfrac{4}{27}\left(sz_1 - \tfrac{3}{2}\right)^3, \end{aligned}\right\} \tag{7}$$

whose right sections are semi-cubic parabolas. Its projecting cylinder upon α_1 is

$$x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 2^{\frac{2}{3}}. \tag{8}$$

By equations (7) the curve is upon the hyperboloid of revolution*

$$x_1^2 + y_1^2 - \tfrac{4}{3} s^2 z_1^2 = 1.$$

Any two successive positions of a plane β, subject to a plane movement, intersect in a straight line an element of its envelope making the same (constant) angle with the plane of movement (σ and σ_1) as β does. Hence, the tangents of the edge of regression of the envelope of β make a constant angle with the plane of movement, i. e., the edge is a curve of constant slope.† The slope of the curve (7) is the reciprocal of s.

The edge of regression has cusps at the four points $\left(2, 0, \frac{3}{2s}\right)$, $\left(0, 2, -\frac{3}{2s}\right)$, $\left(-2, 0, \frac{3}{2s}\right)$, $\left(0, -2, -1, \frac{3}{2s}\right)$, given in order along it. The portions between are convex toward the origin. The projecting cylinders (7) are each tangent to one of the nodal parabolas. Hence, at each cusp of the edge of regression there is a tangent nodal parabola.

Due to the convexity toward the origin of the edge of regression, it is possible to pass a plane between the origin on one side and three of its cusps on the other, so near the latter that the edge of regression pierces the plane in six real points, two in the vicinity of each cusp. The edge of regression is thus of the sixth order, and each piercing point is a cusp on the curve of intersection of the plane and developable. In addition to these cusps, the plane section has four double points, the piercing points of the two nodal parabolas. For the order six, with four double points and six cusps, Plücker's equations give: class four, no inflexions and three double tangents. The multiple points accounted for must be all, since an additional node or cusp would reduce the class to two or one. The two parabolas are the only nodal lines. The value four for the class of the surface, that being the same as the class of its plane sections, is verified by a theorem of Darboux* which states the class of a surface enveloped by a plane having one degree of freedom to be equal to the order of the trajectory of a point subjected to the inverse movement. The inverse gives in this case the limaçon.

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* For this result I am indebted to the "Referee," who has shown that the edge of regression of the envelope of a plane, when the centrodes of the movement are circles, is upon a quadric of revolution.

† The "Referee."

We are now prepared to study the sections of the surface (6) by planes parallel to σ_1. As remarked, they are the same as the envelopes of the straight lines of σ which are parallel to $y = 0$. The section by $z_1 = 0$ is a four-cusped hypocycloid, the cuspidal tangents bisecting the angles between the coordinate axes. Its cusps are at the middle points of the arcs of the hypocycloid

$$x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 2^{\frac{2}{3}}$$

of double its size, which is the projection of the edge of regression. Since $z_1 = 0$ intersects each of the nodal parabolas in two imaginary points, the section has four imaginary double-points. Besides the four real cusps, it has two which are imaginary, the remaining points of intersection of the plane with the edge of regression. As the value of c increases from zero, the four real cusps of the section by $z_1 = c$ move symmetrically away from the axis of y_1 going along the curve $x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 2^{\frac{2}{3}}$ towards its cusps $(\pm 2, 0)$. The curve of section has no real nodes until it becomes tangent to the nodal parabola $x_1^2 - 4sz_1 + 2 = 0$, $y_1 = 0$ for $c = \dfrac{1}{2s}$ when the section has a tac-node at the origin. By further increasing c, the tac-node resolves itself into two real nodes which move along the axis of x_1 away from the origin in opposite directions. For $c = \dfrac{3}{2s}$, two cusps and a node unite at each of the points $(\pm 2, 0)$

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* Koenigs, "Leçons de cinématique, p. 353.

and the curve which has no visible singularities resembles an ellipse whose major axis is on the axis of x_1. The singular points of this curve are the cusps of the edge of regression. For values of c greater than $\frac{3}{2s}$ the sections resemble ellipses, their only real singularities being two conjugate points within and on the axis of x_1. The curves become more and more nearly circular as c increases in value. The curve of section by $z_1 = -c$, when rotated in its plane through a right angle, is the same as the section by $z_1 = c$.

In the case of any plane movement whose centrodes are circles, the characteristic of the envelope of a carried plane through a diameter AB of the moving centrode, is a straight line passing through the center of the centrode and perpendicular to AB. Hence, the projection of the edge of regression upon the plane of movement is the evolute of the cycloid enveloped by AB, i. e., a similar cycloid; and all plane sections of the envelope parallel to the plane of movement are involutes of this projection. For the special movement under consideration, AB envelopes

$$x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 1$$

and its evolute is

$$x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 2^{\frac{2}{3}},$$

which agrees with equation (8). The involutes of this are the sections of (6) by $z_1 =$ const.*

Turning, in conclusion, to the inverse of the ellipsograph, a straight line of σ_1 envelopes a circle in σ whose center is upon c, for it is parallel to a diameter of c_1, and every diameter of c_1 passes through a fixed point of c. The envelope of a carried plane is a cone or a cylinder of revolution.

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* My attention was called to this by the "Referee."